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FACTORWISE RIGIDITY OF PRODUCTS OF PSEUDO-ARCS

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It is proven that every homeomorphism of a product of pseudo-arcs is a composition of a product homeomorphism with a homeomorphism which only permutes the factors.

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product continuum — ε-mapping
product homeomorphism — ε-homeomorphism
homogeneous continuum — pseudo-arc

A *continuum* is a compact, connected Hausdorff space. The interval $[0, 1]$ is denoted by I. A topological space X is *homogeneous* if and only if for each $x, y \in X$ there is a homeomorphism $h: X \to X$ such that $h(x) = y$. *Map* or *mapping* always means continuous function.

If X is a continuum then $\mathscr{H}(X)$, the set of homeomorphisms of X is a topological group which acts continuously on X when $\mathscr{H}(X)$ is given the compact open topology [1]. If Γ is an indexing set, and for each $\gamma \in \Gamma$, X_γ is a continuum, then $\prod_{\gamma \in \Gamma} X_\gamma$ is a *product continuum.* In a product continuum $\prod_{\gamma \in \Gamma} X_\gamma$, a *product homeomorphism* is a homeomorphism of the form $H(\langle x_\gamma \rangle_{\gamma \in \Gamma}) = \langle h_\gamma(x_\gamma) \rangle_{\gamma \in \Gamma}$ where each h_γ is a homeomorphism of X_γ. *Product mapping* is defined similarly. If all the X_γ's are equal to X, $\prod_{\gamma \in \Gamma} X_\gamma$ may be written as X^Γ. If n is a positive integer, $\dot{I}^n$ is the boundary of I^n; that is, the set of those points of I^n with at least one coordinate equal to zero or one.

If X and Y are metric continua and $\varepsilon > 0$, an *ε-map* $f: X \to Y$ is a map such that for each $y \in Y$, $f^{-1}(y)$ has diameter less than ε. If X is metric, an *ε-homeomorphism* $h: X \to X$ is a homeomorphism such that, for each $p \in X$, the distance from p to $h(p)$ is less than ε. The set of ε homeomorphisms is a neighborhood of the identity in $\mathscr{H}(X)$. If each of X_i, $1 \leq i \leq n$, is a metric continuum with metric d_i, the metric

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d on $\prod_{i=1}^n X_i$ is always taken to be the maximum metric; that is, $d(\langle x_i\rangle_{i=1}^n, \langle y_i\rangle_{i=1}^n)$ is the largest element of the set $\{d_i(x_i, y_i)\}_{i=1}^n$. With this convention, it follows that a product mapping is an ε-map if and only if each of the factor maps is an ε-map, and a product homeomorphism is an ε-homeomorphism if and only if each of its factor maps is an ε-homeomorphism.

A map $f: X \to I^n$ is *essential* provided there does not exist a nonsurjective mapping $g: X \to I^n$ such that $g|f^{-1}(\dot{I}^n) = f|f^{-1}(\dot{I}^n)$. If X and Y are continua, a mapping $f: X \to Y$ is *weakly confluent* if and only if, for every subcontinuum $M \subseteq Y$, there exists a subcontinuum $W \subseteq X$ such that $f(W) = M$.

Lemma 1. *Every essential map of a metric continuum X onto I^n is weakly confluent.*

Proof. Mazurkiewicz [13] proved this for the case $n = 2$ and said in a footnote that it was true for all n. Howard Cook has pointed out that the same proof works for all n. $\square$

Lemma 2. *If $\{X_i\}_{i=1}^n$ is a finite family of metric continua and for each i, $f_i: X_i \to I$ is an onto mapping, then the product mapping*

$$F = \prod_{i=1}^n f_i: \prod_{i=1}^n X_i \to I^n$$

is essential.

Proof. This is a fairly straightforward application of the Brouwer fixed point theorem; it may well be a standard exercise in dimension theory. Let each X_i be embedded into a copy Q_i of the Hilbert cube. If F is not essential, there is a map $g: \prod_{i=1}^n X_i \to \dot{I}^n$ which agrees with F on $F^{-1}(\dot{I}^n)$. Then, since $\dot{I}^n$ is an ANR, g extends continuously to a product of closed neighborhoods W_i of X_i in Q_i. Call this extension $G: \prod_{i=1}^n W_i \to \dot{I}^n$.

Let $\delta > 0$ be such that whenever $x, y \in \prod_{i=1}^n W_i$ and $d(x, y) < \delta$, then $d(G(x), G(y)) < \frac{1}{2}$. Assume, with no loss of generality, that each point of each W_i is at a distance less than δ from some point of X_i. Choose $a_i, b_i \in X_i$ such that $f_i(a_i) = 0$ while $f_i(b_i) = 1$. Let $h_i: I \to W_i$ be a path such that $h_i(1) = a_i$ and $h_i(0) = b_i$. Then

$$G \circ \prod_{i=1}^n h_i: I^n \to \dot{I}^n \subseteq I^n$$

is a fixed point free map, a contradiction; and the proof is complete.

A continuum X is *decomposable* provided there exist proper subcontinua $A, B \subseteq X$ such that $X = A \cup B$; otherwise X is *indecomposable*. Equivalently, X is indecomposable if and only if every proper subcontinuum of X is nowhere dense. For a nondegenerate indecomposable continuum X, p and q belong to the same *composant* of X provided some proper subcontinuum W of X contains both p and q. This is easily seen to be an equivalence relation on X, and for X metric, every equivalence class (composant) is known to be a dense first category F_σ set [12]. If p and q lie in different composants of X, then X is irreducible between them. X is *hereditarily indecomposable* provided every subcontinuum of X is indecomposable. Note that it is immediate from this definition that if X is hereditarily indecomposable and A, B are subcontinua of X with $A \cap B \neq \emptyset$, then either $A \subseteq B$ or $B \subseteq A$.

A metric continuum is *chainable* or *arc-like* if and only if, for each $\varepsilon > 0$, there exists an ε-map $f: X \to I$. A metric continuum which is both chainable and hereditarily indecomposable is called a *pseudo-arc*. Clearly each nondegenerate subcontinuum of a pseudo-arc is a pseudo-arc. For background material on pseudo-arcs, see for example [2, 3, 4, 5, 6, 7, 10, 11, 15, 16].

The letter P, with or without subscripts, will henceforth denote a fixed pseudo-arc with a given metric ρ.

For any set Γ, and any space X, the group of product homeomorphisms of X^Γ will be denoted $\mathscr{G}(X^\Gamma)$. A homeomorphism $h \in \mathscr{H}(X^\Gamma)$ is *factor preserving* if and only if $h = \theta \circ H$ where $H \in \mathscr{G}(X^\Gamma)$ and θ only permutes the factors of X^Γ. If every self-homeomorphism of X^Γ is factor-preserving, X^Γ is *factorwise rigid*.

M is the Menger universal curve. K. Kuperberg, W. Kuperberg, and Transue [8] proved that $M \times M$ is factorwise rigid. The second author [17] extended this result to arbitrary products of M or of Sierpinski universal curves. Lewis [9, problem 60] has asked whether P has the same property. Lysko and Bellamy [3] have given a proof that this is true for the product $P \times P$, using techniques that do not directly generalize to larger products. In the present article, this result is extended to all products of pseudo-arcs.

The next four lemmas summarize the principal facts about pseudo-arcs which will be used here.

Lemma 3 [5] or [16]. *P is homogeneous.*

Lemma 4 [6]. *P has the fixed point property.*

Lemma 5 [10]. *Given any homeomorphism $h: P \to P$ and any $\varepsilon > 0$ there is a finite sequence $h_1, h_2, \ldots, h_n$ of ε-homeomorphisms of P whose composition $h_1 \circ h_2 \circ \cdots \circ h_n$ is h.*

Lemma 6. *Given any $\varepsilon > 0$ and any p, q which belong to different composants of P, there exists an ε-map $f: P \to 1$ such that $f^{-1}(0) = \{p\}$ and $f^{-1}(1) = \{q\}$.*

Proof. This follows easily from Theorem 1 of [4, p. 44] and the proof thereof. $\square$

In a product $\prod_{\gamma \in \Gamma} X_\gamma$, a set obtained by fixing a single coordinate will be called a *slice*. In particular, the set

$$L(\alpha, p) = \left\{ \langle x_\gamma \rangle_{\gamma \in \Gamma} \in \prod_{\gamma \in \Gamma} X_\gamma \,\middle|\, x_\alpha = p \right\}$$

will be called an *α-slice*, or, if the point p is needed, the *p-α slice*. The same notation will be used if $y=\langle y_\gamma\rangle_{\gamma\in\Gamma}\in\prod_{\gamma\in\Gamma} X_\gamma$; in this case, $L(\alpha, y)=\{\langle x_\gamma\rangle_{\gamma\in\Gamma}\in\prod_{\gamma\in\Gamma} X_\gamma \mid x_\alpha = y_\alpha\}$. Note that for fixed α, distinct α-slices are disjoint; and if one slice contains another slice, the two are equal. In I^Γ, the $0-\alpha$ and $1-\alpha$ slices are called the opposite α-faces of I^Γ.

Lemma 7. *A homeomorphism h of a product continuum $\prod_{\gamma\in\Gamma} X_\gamma$ is a product homeomorphism if and only if, for every α, the image under h of every α-slice is an α-slice.*

Proof. Define $h_\alpha : X_\alpha \to X_\alpha$ by $h_\alpha(x)=y$ where $h(L(\alpha, x))=L(\alpha, y)$. It is clear that $h=\prod_{\alpha\in\Gamma} h_\alpha$. □

Henceforth, $\pi_\alpha : \prod_{\gamma\in\Gamma} X_\gamma \to X_\alpha$ will always denote the projection map, with the exception that, in I^Γ, $\hat{\pi}_\alpha : I^\Gamma \to I_\alpha = I$ will denote the projection.

Lemma 8. *Let X be a nondegenerate continuum. A homeomorphism $h\in\mathscr{H}(X^\Gamma)$ is factor preserving if and only if the image of every slice under h is a slice.*

Proof. Fix $\alpha\in\Gamma$, and define $A(\beta)=\{p\in X \mid h(L(\alpha, p))$ is a β-slice$\}$. Note that each $A(\beta)$ is open in X. To see this, suppose $x\in A(\beta)$, and $h(L(\alpha, x))=L(\beta, y)$. For any U open in X such that $y\in U$ but $U\neq X$, there is some open set $\hat{V}$ in $\prod_{\gamma\in\Gamma} X_\gamma$ such that $L(\alpha, x)\subseteq\hat{V}$ and $\pi_\beta\circ h(\hat{V})\subseteq U$. Then using compactness, there is an open set V in X containing x such that

$$\left\{\langle z_\gamma\rangle_{\gamma\in\Gamma}\in \prod_{\gamma\in\Gamma} X_\gamma \,\middle|\, z_\alpha\in V\right\}\subseteq\hat{V}.$$

Thus, if $w\in V$, $\pi_\beta\circ h(L(\alpha, w))\neq X$, and $h(L(\alpha, w))$ is a β-slice. Thus $x\in V\subseteq A(\beta)$, so that $A(\beta)$ is open. Hence, $\{A(\beta)\}_{\beta\in\Gamma}$ is a covering of X by pairwise disjoint open sets. Since X is connected, $A(\beta)=X$ for some β; denote this β by $\phi(\alpha)$. Then $\phi:\Gamma\to\Gamma$ is a function.

To prove that the image of every slice under h^{-1} is a slice, suppose $L(\beta, y)$ is a β-slice. Since $\{h^{-1}(y)\}=\bigcap\{L(\alpha, h^{-1}(y)) \mid \alpha\in\Gamma\}$, it follows that $\{y\}=\bigcap\{h(L(\alpha, h^{-1}(y))) \mid \alpha\in\Gamma\}$. Let $\hat{y}$ be a point differing from y only in the βth coordinate. Then since $\hat{y}\notin\bigcap\{h(L(\alpha, h^{-1}(y))) \mid \alpha\in\Gamma\}$, there is an $\hat{\alpha}\in\Gamma$ such that $\hat{y}\notin h(L(\hat{\alpha}, h^{-1}(y)))$. Then, since $h(L(\hat{\alpha}, h^{-1}(y)))$ is a slice, it must be a β-slice; in particular, $h(L(\hat{\alpha}, h^{-1}(y)))=L(\beta, y)$. Thus, $h^{-1}(L(\beta, y))=L(\hat{\alpha}, h^{-1}(y))$, as required. Hence, ϕ^{-1} is a function as well, so that $\phi:\Gamma\to\Gamma$ is bijective. This implies that h is factor-preserving, being a product homeomorphism followed by the permutation ϕ on the factors.

The converse is clear, and the proof is complete. □

The set function T is defined on the subsets of a continuum X as follows: $T(A)$ is that subset of X such that for $x\in X$, $x\notin T(A)$ if and only if x has a closed, connected neighborhood W such that $W \cap A = \emptyset$. A substantial body of literature exists on the set function T; the interested reader is referred to the articles in Sections II and V of [14] and the bibliographies thereof. The next three lemmas give the only facts about T which are needed here.

Lemma 9. *If $\{X_\gamma\}_{\gamma\in\Gamma}$ is a family of indecomposable continua and $L \subseteq \prod_{\gamma\in\Gamma} X_\gamma$ is a slice, then $T(L) = \prod_{\gamma\in\Gamma} X_\gamma$.*

Proof. If W is a subcontinuum of $\prod_{\gamma\in\Gamma} X_\gamma$ and W has nonempty interior, then its projection to every X_γ must be onto. Thus W meets every slice. $\square$

Lemma 10. *If Γ is nondegenerate and $\prod_{\gamma\in\Gamma} X_\gamma$ is a product of nondegenerate continua and A is a proper subset of a slice in $\prod_{\gamma\in\Gamma} X_\gamma$, then $T(A) \neq \prod_{\gamma\in\Gamma} X_\gamma$.*

Proof. Assume $A \subseteq L(\alpha, p)$ and let $\langle a_\gamma\rangle_{\gamma\in\Gamma} \in L(\alpha, p) - A$. Let U be a closed set with interior in X_α such that $p \notin U$. Then

$$\{\langle x_\gamma\rangle_{\gamma\in\Gamma} \mid x_\alpha \in U \text{ or for every } \gamma \neq \alpha,\ x_\gamma = a_\gamma\}$$

is a continuum with nonempty interior which misses A. (This is a classical argument of F.B. Jones.) $\square$

Lemma 11. *If X is a continuum, $A \subseteq X$, and $h: X \to X$ is a homeomorphism, then $T(h(A)) = h(T(A))$.*

Proof. This is clear. $\square$

If $B \subseteq \Gamma$ then $Q_B: \prod_{\gamma\in\Gamma} X_\gamma \to \prod_{\gamma\in B} X_\gamma$ is the projection, $Q_B(\langle x_\gamma\rangle_{\gamma\in\Gamma}) = \langle x_\gamma\rangle_{\gamma\in B}$, with the exception that $\hat{Q}_B$ denotes this projection in I^Γ.

Lemma 12. *Let L be a slice in P^Γ. Suppose $h \in \mathcal{H}(P^\Gamma)$ and that for some α, $\pi_\alpha \circ h(L) \neq P$. Let $f_\gamma: P \to I$ be an onto map for each $\gamma \in \Gamma$ and assume that $f_\alpha^{-1}(0)$ and $f_\alpha^{-1}(1)$ are singletons lying in different composants of P and missing $\pi_\alpha \circ h(L)$. Let $F = \prod_{\gamma\in\Gamma} f_\gamma: P^\Gamma \to I^\Gamma$. Then $F(h(L))$ separates I^Γ between its two α-faces. (That is, if C is a continuum in I^Γ which intersects both $\{x \in I^\Gamma \mid x_\alpha = 0\}$ and $\{x \in I^\Gamma \mid x_\alpha = 1\}$, then $F(h(L)) \cap C \neq \emptyset$.)*

Proof. Let $p_0 = f_\alpha^{-1}(0)$ and $p_1 = f_\alpha^{-1}(1)$. Suppose that there is a continuum $A \subseteq I^\Gamma$ with points a_0 and a_1 on the 0- and 1- α-faces, respectively, such that $A \cap F(h(L)) = \emptyset$. Then there is a finite subset B of Γ containing α such that $\hat{Q}_B(A) \cap \hat{Q}_B(F(h(L))) = \emptyset$. Now since $F_B = \prod_{\gamma\in B} f_\gamma: P^B \to I^B$ is weakly confluent, there is some continuum $K \subseteq P^B$ such that $F_B(K) = \hat{Q}_B(A)$. There are points b_0 and b_1 in K such that $F_B(b_0) = \hat{Q}_B(a_0)$ and $F_B(b_1) = \hat{Q}_B(a_1)$. But then the αth coordinate of b_0 is p_0 and the αth coordinate of b_1 is p_1. It follows that $\pi_\alpha(Q_B^{-1}(K)) = P_\alpha = P$, since Q_B is monotone and no proper subcontinuum of P contains $\{p_0, p_1\}$. Suppose U is a closed neighborhood of p_0 in P_α missing $\pi_\alpha \circ h(L)$, and let $M = \{x \in P^\Gamma \mid x_\alpha \in U\}$. Then $M \cup Q_B^{-1}(K)$ is a continuum in P^Γ with nonempty interior which misses $h(L)$. But this can't be, since $T(h(L)) = P^\Gamma$. This contradiction completes the proof. □

Lemma 13. *Suppose $L \subseteq P^\Gamma$ is a slice and $h \in \mathscr{H}(P^\Gamma)$ and $h(L)$ is not a slice. Then for every $\gamma \in \Gamma$, $\pi_\gamma \circ h(L) = P$.*

Proof. Suppose for some $\alpha \in \Gamma$, $\pi_\alpha(h(L)) \neq P_\alpha$. Let $p_0, p_1 \in P_\alpha - \pi_\alpha(h(L))$, and assume that p_0 and p_1 belong to distinct composants of P_α, and neither belongs to the composant containing $\pi_\alpha(h(L))$. Without loss of generality, assume L is a β-slice, and let $\langle a_\gamma \rangle_{\gamma \in \Gamma} \in L$.

Let $\langle W_k \rangle_{k=1}^\infty$ be a decreasing sequence of nondegenerate subcontinua of P_β whose intersection is $\{a_\beta\}$. Let $W(k) = W_k \times \{\langle a_\gamma \rangle_{\gamma \neq \beta}\}$. By hypothesis, $h(L)$ is not a slice. If $h(L)$ is a proper subset of a slice, then $T(h(L)) \neq P^\Gamma$ by Lemma 10, while by Lemmas 9 and 11, $T(h(L)) = P^\Gamma$. Therefore, $\pi_\alpha(h(L))$ is nondegenerate, and it follows that, for some m, $\pi_\alpha(h(W(m))) \subseteq \pi_\alpha(h(L))$. Choose $b \in W_m$ sufficiently close to a_β that $\pi_\alpha(h(L(\beta, b))) \neq P_\alpha$, and let $J = L(\beta, b)$. Then $\pi_\alpha(h(J)) \cap \pi_\alpha(h(L)) \neq \emptyset$, so that either $\pi_\alpha(h(J)) \subseteq \pi_\alpha(h(L))$ or $\pi_\alpha(h(L)) \subseteq \pi_\alpha(h(J))$. Assume the former; the two cases are identical. There is a finite subset B of Γ, containing α, such that $Q_B(h(L)) \cap Q_B(h(J)) = \emptyset$. Since $Q_B(h(L))$ and $Q_B(h(J))$ are subsets of $\prod_{\gamma \in B} P_\gamma$, there is $\varepsilon > 0$ such that ε is less than the distance from $Q_B(h(J))$ to $Q_B(h(L))$ (with respect to the max metric on $\prod_{\gamma \in B} P_\gamma$). For each $\gamma \in \Gamma$ let $f_\gamma : P \to I$ be an ε-map, and assume that $f_\alpha^{-1}(0) = p_0$ and $f_\alpha^{-1}(1) = p_1$. Then $F = \prod_{\gamma \in \Gamma} f_\gamma : P^\Gamma \to I^\Gamma$ is a continuous map, and $F_B = \prod_{\gamma \in B} f_\gamma : P^B \to I^B$ is an ε-map. Thus, $F_B(Q_B(h(J))) \cap F_B(Q_B(h(L))) = \emptyset$, and since $F_B(Q_B(h(J))) = \hat{Q}_B(F(h(J)))$ and $F_B(Q_B(h(L))) = \hat{Q}_B(F(h(L)))$, it follows that $\hat{Q}_B(F(h(J))) \cap \hat{Q}_B(F(h(L))) = \emptyset$ and that $F(h(J)) \cap F(h(L)) = \emptyset$. Furthermore, by Lemma 12, $F(h(J))$ separates I^Γ between its two α-faces. (Neither $F(h(J))$ nor $F(h(L))$ meets an α-face by composant choices for p_0 and p_1.) However, $\hat{\pi}_\alpha(F(h(J))) = f_\alpha(\pi_\alpha(h(J)))$, and since $\pi_\alpha(h(J)) \subseteq \pi_\alpha(h(L))$, it follows that $f_\alpha(\pi_\alpha(h(J))) \subseteq f_\alpha(\pi_\alpha(h(L)))$, and thus $\hat{\pi}_\alpha(F(h(J))) \subseteq \hat{\pi}_\alpha(F(h(L)))$. Let $\langle x_\gamma \rangle_{\gamma \in \Gamma}$ and $\langle y_\gamma \rangle_{\gamma \in \Gamma}$ be points in $F(h(L))$ with smallest and largest αth coordinates, respectively. Let

$$A = \{\langle t_\gamma \rangle_{\gamma \in \Gamma} \in I^\Gamma \mid \text{for each } \gamma \neq \alpha,\ t_\gamma = x_\gamma, \text{ and } 0 \leq t_\alpha \leq x_\alpha\},$$

$$B = \{\langle t_\gamma \rangle_{\gamma \in \Gamma} \in I^\Gamma \mid \text{for each } \gamma \neq \alpha,\ t_\gamma = y_\gamma, \text{ and } y_\alpha \leq t_\alpha \leq 1\}.$$

Then, each of A and B is a straight line segment meeting $F(h(L))$, and $A \cup B \cup F(h(L))$ is a continuum in I^Γ joining the two α-faces but missing $F(h(J))$. But $F(h(J))$ separates I^Γ between these two faces, and the proof is done by contradiction. □

Lemma 14. *Suppose $h \in \mathscr{H}(P^\Gamma)$, L is an α-slice and $\pi_\alpha \circ h(L) \neq P_\alpha$. Then $h(L)$ is an α-slice.*

Proof. The proof of Lemma 13 gives this result, too. $\square$

Lemma 15. *Suppose $h \in \mathcal{H}(P^\Gamma)$, $\alpha \in \Gamma$, and there is an α-slice L such that $\pi_\alpha \circ h(L) \neq P_\alpha$. Then for each α-slice L', $h(L')$ is an α-slice.*

Proof. Let $A = \{x \in P \mid h(L(x, \alpha))$ is an α-slice$\}$, and $B = \{x \in P \mid h(L(x, \alpha))$ is not an α-slice$\}$. Then $A \neq \emptyset$, since if $z \in L$, $L = L(z_\alpha, \alpha)$ and $h(L)$ is an α-slice. If $B \neq \emptyset$, $A \cup B = P$, and $A \cap B = \emptyset$.

Let $a \in A$ and define x by $h(L(a, \alpha)) = L(x, \alpha)$. Suppose $B \neq \emptyset$ and let $b \in B$. Then $h(L(b, \alpha))$ is not an α-slice, and so by Lemma 13, $\pi_\alpha(h(L(b, \alpha))) = P_\alpha$; in particular, $x \in \pi_\alpha(h(L(b, \alpha)))$, so that $h(L(b, \alpha)) \cap L(x, \alpha) \neq \emptyset$; that is $h(L(b, \alpha)) \cap h(L(a, \alpha)) \neq \emptyset$. Consequently $L(b, \alpha) \cap L(a, \alpha) \neq \emptyset$. Thus $a = b$, which is impossible. Hence, $B = \emptyset$ and $A = P$. $\square$

Theorem 1. *For each finite set B in Γ, let $G(B) = \{h \in \mathcal{H}(P^\Gamma) \mid$ for each $\alpha \in B$, if L is an α-slice, $h(L)$ is an α-slice$\}$. Then $G(B)$ is an open-closed subgroup of $\mathcal{H}(P^\Gamma)$.*

Proof. Clearly $G(B)$ is a group. Also, $G(B) = \bigcap_{\alpha \in B} G(\{\alpha\})$, and so, to show that $G(B)$ is open, all that needs to be shown is that $G(\{\alpha\})$ is open for each α. Fix α. For convenience, let $G(\{\alpha\}) = G$.

Recall that the topology on $\mathcal{H}(P^\Gamma)$ is the compact-open topology. If C is a compact subset of P^Γ and U is open in P^Γ, then $\langle C, U\rangle = \{h \in \mathcal{H}(P^\Gamma) \mid h(C) \subseteq U\}$ is a sub-basic open set in $\mathcal{H}(P^\Gamma)$. To see that some open set containing 1 is in G, suppose $U(\alpha)$ and $V(\alpha)$ are open in P such that $U(\alpha) \cup V(\alpha) = P$, but $\overline{U(\alpha)} \neq P$ and $\overline{V(\alpha)} \neq P$. Then find $H(\alpha)$ and $K(\alpha)$ such that $H(\alpha)$ and $K(\alpha)$ are closed subsets of P such that $H(\alpha) \cup K(\alpha) = P$, $H(\alpha) \subseteq U(\alpha)$ and $K(\alpha) \subseteq V(\alpha)$. Let $U = \{x \in P^\Gamma \mid x_\alpha \in U(\alpha)\}$, $V = \{x \in P^\Gamma \mid x_\alpha \in V(\alpha)\}$, $H = \{x \in P^\Gamma \mid x_\alpha \in H(\alpha)\}$ and $K = \{x \in P^\Gamma \mid x_\alpha \in K(\alpha)\}$. Then $1 \in \langle H, U\rangle \cap \langle K, V\rangle$, which is open in $\mathcal{H}(P^\Gamma)$. But also, $\langle H, U\rangle \cap \langle K, V\rangle \subseteq G$, by Lemma 14. Thus, G is open. But then G is closed, too, for all open subgroups are closed. $\square$

Theorem 2. *Recall that $\mathcal{G}(P^\Gamma)$ denotes the set of product homeomorphisms in $\mathcal{H}(P^\Gamma)$. Then $\mathcal{G}(P^\Gamma) = \bigcap_{\alpha \in \Gamma} G(\{\alpha\})$ and $\mathcal{G}(P^\Gamma)$ is a closed, normal subgroup of $\mathcal{H}(P^\Gamma)$.*

Proof. Clearly $\mathcal{G}(P^\Gamma)$ is a closed subgroup of $\mathcal{H}(P^\Gamma)$, since $\mathcal{G}(P^\Gamma) = \bigcap_{\alpha \in \Gamma} G(\{\alpha\})$.

Suppose $h \in \mathcal{G}(P^\Gamma)$ and $g \in \mathcal{H}(P^\Gamma)$. Consider $g \circ h \circ g^{-1}$. Write $h = \prod_{\alpha \in \Gamma} h_\alpha$ where, for $\alpha \in \Gamma$, $h_\alpha \in \mathcal{H}(P)$. For B a finite set of Γ, let $h(B) = \prod_{\alpha \in \Gamma} h(B)_\alpha$ where $h(B)_\alpha = 1$ $(\in \mathcal{H}(P))$ for $\alpha \notin B$, and $h(B)_\alpha = h_\alpha$ for $\alpha \in B$. Further, let $h(\tilde{B}) = \prod_{\alpha \in \Gamma} h(\tilde{B})_\alpha$ where $h(\tilde{B})_\alpha = 1$ $(\in \mathcal{H}(P))$ for $\alpha \in B$, and $h(\tilde{B})_\alpha = h_\alpha$ for $\alpha \notin B$.

If $1 \in V$, an open set in $\mathcal{H}(P^\Gamma)$, there is an open set U in $\mathcal{H}(P^\Gamma)$ such that $1 \in U$ and if $f \in U$, then $g \circ f \circ g^{-1} \in V$. (This is because of the continuity of the group operations.) To see that if $\alpha \in \Gamma$, then $g \circ h \circ g^{-1} \in G(\{\alpha\})$, fix α. Since $G(\{\alpha\})$ is open, there is a basic open set U containing 1 such that $g \circ U \circ g^{-1} \subseteq G(\{\alpha\})$. Write

$U=\bigcap_{i=1}^{n}\langle C(i), U(i)\rangle$, i.e., as an intersection of sub-basic sets. Since $1\in U$, $C(i)\subseteq U(i)$ for each i. For each i, there are finite collections $\{V(i,1),\ldots, V(i,m(i))\}$ and $\{V'(i,1),\ldots, V'(i,m(i))\}$ of basic open sets of P^{Γ} such that

$$C(i)\subseteq\bigcup_{j=1}^{m(i)} V'(i,j)\subseteq\bigcup_{j=1}^{m(i)}\overline{V'(i,j)}\subseteq\bigcup_{j=1}^{m(i)} V(i,j)\subseteq U(i),$$

and $\overline{V'(i,j)}\subseteq V(i,j)$ for each $j\leq m(i)$. Now $C(i)=\bigcup_{j=1}^{m(i)}(C(i)\cap\overline{V'(i,j)})$ and for $j\leq m(i)$, $D(i,j)=C(i)\cap\overline{V'(i,j)}\subseteq V(i,j)$.

Note that if $\mathcal{O}$ is a basic open set in P^{Γ} and C is a closed set in $\mathcal{O}$, then there is a finite set B_i in Γ such that $h(\tilde{B}_i)\in\langle C,\mathcal{O}\rangle$. Thus, there is a finite set B_i in Γ such that

$$h(\tilde{B}_i)\in\bigcap_{j=1}^{m(i)}\langle D(i,j), V(i,j)\rangle\subseteq\langle C(i), U(i)\rangle.$$

Let $B=\bigcup_{i=1}^{n} B_i$. Then

$$h(\tilde{B})\in\bigcap_{i=1}^{n}\left(\bigcap_{j=1}^{m(i)}\langle D(i,j), V(i,j)\rangle\right)\subseteq\bigcap_{i=1}^{n}\langle C(i), U(i)\rangle=U,$$

and $g\circ h(\tilde{B})\circ g^{-1}\in G(\{\alpha\})$.

For $\beta\in B$, Lemma 5 can be used to write h_β as a finite composition $h_\beta(1)\circ h_\beta(2)\circ\cdots\circ h_\beta(k)$ of homeomorphisms of P such that if $h_\beta(i)=\prod_{\gamma\in\Gamma}f_\gamma$ where $f_\gamma=1\in\mathscr{H}(P)$ if $\gamma\neq\beta$, and $f_\beta=h_\beta(i)$, then $g\circ\hat{h}_\beta(i)\circ g^{-1}\in G(\{\alpha\})$ and $g\circ h(\{\beta\})\circ g^{-1}=(g\circ\hat{h}_\beta(1)\circ g^{-1})\circ(g\circ\hat{h}_\beta(2)\circ g^{-1})\circ\cdots\circ(g\circ\hat{h}_\beta(k)\circ g^{-1})\in G(\{\alpha\})$. But then $g\circ h(B)\circ g^{-1}\in G(\{\alpha\})$, since $h(B)$ is a composition of the $h(\{\beta\})$'s for $\beta\in B$ (in any order; they commute) and $h=h(B)\circ h(\tilde{B})$ so that $g\circ h\circ g^{-1}=(g\circ h(B)\circ g^{-1})\circ(g\circ h(\tilde{B})\circ g^{-1})\in G(\{\alpha\})$. This completes the proof. □

Theorem 3. *Every homeomorphism of a product of pseudo-arcs is factor preserving; that is, products of pseudo-arcs are factorwise rigid.*

Proof. It suffices, by Lemma 8, to prove that the image of every slice under any $h\in\mathscr{H}(P^{\Gamma})$ is a slice. Suppose to the contrary that there exists a slice $L\subseteq P^{\Gamma}$ and an $h\in\mathscr{H}(P^{\Gamma})$ such that $h(L)$ is not a slice. By Lemma 13, for every γ, $\pi_\gamma(h(L))=P$. Let $\langle p_\gamma\rangle_{\gamma\in\Gamma}\in h(L)$, and let $\langle q_\gamma\rangle_{\gamma\in\Gamma}$ be any point in P^{Γ} differing from $\langle p_\gamma\rangle_{\gamma\in\Gamma}$ in only one coordinate; that is, for some α, $p_\alpha\neq q_\alpha$, but $p_\gamma=q_\gamma$ for $\gamma\neq\alpha$. Let $g_\alpha: P\to P$ be a homeomorphism such that $g_\alpha(p_\alpha)=q_\alpha$ and let a_α be a fixed point of g_α. For $\gamma\neq\alpha$, let g_γ be the identity map, and $g=\prod_{\gamma\in\Gamma}g_\gamma$ be the product homeomorphism. Since $\pi_\alpha(h(L))=P$, in particular, $a_\alpha\in\pi_\alpha(h(L))$, and so there exists a point $a=\langle a_\gamma\rangle_{\gamma\in\Gamma}\in h(L)$ such that $\pi_\alpha(a)=a_\alpha$. Since, for each γ, a_γ is a fixed point of g_γ, a is a fixed point of g. Now $g\in\mathscr{G}(P^{\Gamma})$, and so $h^{-1}\circ g\circ h\in\mathscr{G}(P^{\Gamma})$ also. Consequently, $h^{-1}\circ g\circ h(L)$ is a slice in the same direction as L. However, since $a\in h(L)$, there exists $x\in L$ such that $h(x)=a$. Then $g(h(x))=g(a)=a$, and $h^{-1}\circ g\circ h(x)=x$. Therefore, $h^{-1}\circ g\circ h(L)\cap L\neq\emptyset$, so that $h^{-1}\circ g\circ h(L)=L$. Thus, since $\langle q_\gamma\rangle_{\gamma\in\Gamma}=g(\langle p_\gamma\rangle_{\gamma\in\Gamma})\in g(h(L))$; $h^{-1}(\langle q_\gamma\rangle_{\gamma\in\Gamma})\in h^{-1}\circ g\circ h(L)=L$, and so $\langle q_\gamma\rangle_{\gamma\in\Gamma}\in h(L)$. Thus, every point of P^{Γ}, which differs in only one coordinate from a point of $h(L)$, itself belongs to $h(L)$. However, if B is a finite subset of Γ, $\langle p_\gamma\rangle_{\gamma\in\Gamma}\in h(L)$ and $\langle s_\gamma\rangle_{\gamma\in\Gamma}\in P^{\Gamma}$ such that $s_\gamma = p_\gamma$ for each $\gamma\notin B$, there is a finite sequence $\{x(k)\}_{k=0}^{n}$ of points of P^{Γ} such that $x(0)=\langle p_\gamma\rangle_{\gamma\in\Gamma}$; $x(n)=\langle s_\gamma\rangle_{\gamma\in\Gamma}$, and each $x(k-1)$ differs from $x(k)$ only in one coordinate. Thus if $P'=\{z\in P^{\Gamma}\mid z_\gamma=p_\gamma \text{ for } \gamma\notin B\}$, $P'\subseteq h(L)$. But then $h(L)=P^{\Gamma}$, for $h(L)$ is a closed set. This is a contradiction, since $h(L)$ is a proper subcontinuum of P^{Γ}. $\square$

It is peculiar that homogeneity plays such a strong role in this proof. Is every (finite) product of hereditarily indecomposable continua factorwise rigid?

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